

Practice proofs #4

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We explicitly construct a bijection mapping from the unit interval to the whole real line.

Theorem 1. Define $f : (0, 1) \rightarrow \mathbb{R}$ by

$$f(x) = \frac{1}{1-x} - \frac{1}{x}.$$

The function f is a bijection.

Proof. We begin by proving that f is injective. The derivative of f is

$$\frac{1}{(1-x)^2} + \frac{1}{x^2}.$$

Since $(1-x)^2$ and x^2 must always be greater than 0 in the interval $(0,1)$, the derivative of f is always positive. Thus, f is strictly increasing in the interval $(0,1)$. Therefore, f is an injection.

To see that f is surjective, consider that for large $M > 0$,

$$\begin{aligned} f\left(\frac{1}{M}\right) &= \frac{1}{1-\frac{1}{M}} - \frac{1}{\frac{1}{M}} \\ &= \frac{1}{1-\frac{1}{M}} - M. \end{aligned}$$

Since, a large M will always be greater than $\frac{1}{1-\frac{1}{M}}$, $f(\frac{1}{M})$ will be a large negative number.

Moreover, consider

$$\begin{aligned} f\left(1 - \frac{1}{M}\right) &= \frac{1}{1-1+\frac{1}{M}} - \frac{1}{1-\frac{1}{M}} \\ &= M - \frac{1}{1-\frac{1}{M}}. \end{aligned}$$

Since, a large M will always be greater than $\frac{1}{1-\frac{1}{M}}$, $f(1 - \frac{1}{M})$ will be a large positive number.

Thus, using the intermediate value theorem, since, for some M , $f(\frac{1}{M}) < y < f(1 - \frac{1}{M})$, the required x must exist. Therefore, f is surjective.

Thus, since f is both injective and surjective, f is a bijection. $\square$