

Practice proofs #8

Anonymous

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We establish that the Cauchy “equivalence” is indeed an equivalence relation.

Theorem 1. *The relation $\sim$ on $C_{\mathbb{Q}}$ given by*

$$(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}} \quad \text{if} \quad (a_n - b_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

is an equivalence relation.

Proof. For a relation to be an equivalence relation, it has to be reflexive, transitive and symmetric. By the definition of limit from Definition 5.1 in reading 11, $(a_n - b_n)_{n \in \mathbb{N}}$ has limit 0 if for any $\epsilon > 0$, there exists a natural number $N = N(\epsilon)$ such that for all $n > N$, $|a_n - b_n| < \epsilon$.

The relation $\sim$ is reflexive if for all $(a_n)_{n \in \mathbb{N}} \in C_{\mathbb{Q}}$, $(a_n)_{n \in \mathbb{N}} \sim (a_n)_{n \in \mathbb{N}}$. This implies $(a_n - a_n) \rightarrow 0$ as $n \rightarrow \infty$. $(a_n - a_n) = 0$, therefore the relation is reflexive.

The relation $\sim$ is symmetric if $(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$ is implied if and only if $(b_n)_{n \in \mathbb{N}} \sim (a_n)_{n \in \mathbb{N}}$. $(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$ implies that $(a_n - b_n) \rightarrow 0$ as $n \rightarrow \infty$. $(b_n)_{n \in \mathbb{N}} \sim (a_n)_{n \in \mathbb{N}}$ implies that $(b_n - a_n) \rightarrow 0$ as $n \rightarrow \infty$. $|a_n - b_n| = |-(b_n - a_n)| = |b_n - a_n|$. As $|b_n - a_n|$ implies $|a_n - b_n|$ and vice versa, then it is symmetric.

The relation $\sim$ is transitive if $(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}} \sim (c_n)_{n \in \mathbb{N}}$ implies $(a_n)_{n \in \mathbb{N}} \sim (c_n)_{n \in \mathbb{N}}$. Suppose the sequences $(a_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ both converge to the limit 0 as n approaches 0. Then the relation between any two is transitive, and therefore it is an equivalence relation. $\square$