

Linearized KdV on the line with a metric graph defect

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University of Wollongong Symposium in honour of Philip Broadbridge
Wollongong NSW, Australia



KdV on the line with a defect

Rightward propagating waves in shallow water

$$u_t(x, t) - u_{xxx} + u_x + 6uu_x = 0 \quad x \in \text{edges}, \quad t > 0, \quad (1.\text{PDE})$$

$$u(x, 0) = U(x) \quad x \in \text{edges}, \quad (1.\text{IC})$$

$$\text{appropriate vertex conditions}, \quad t \geq 0. \quad (1.\text{VC})$$

Here (1.VC) represents a complex defect in the channel.



Mismatch condition between left and right limits at $x = 0$.

KdV on the line with a defect

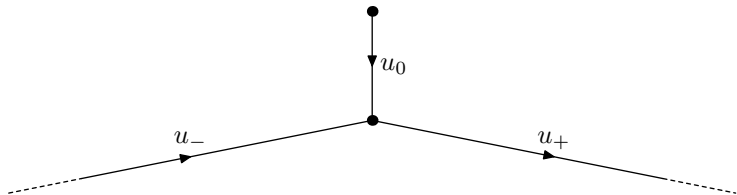
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Information is pushed into the line system at $x = 0$.

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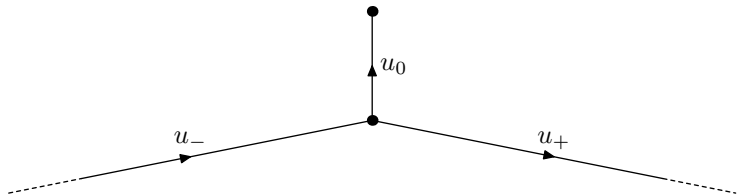
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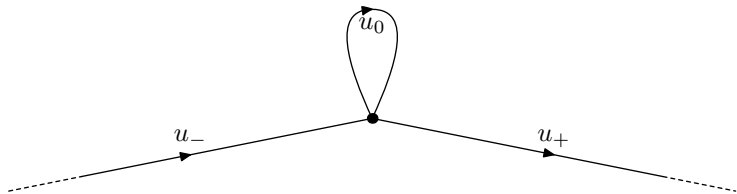
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Related wellposedness results:

Wellposedness for star graph problems with all edges semiinfinite:

M Cavalcante 2018.

Wellposedness for problems with all edges finite:

(for BBM) J Bona, R Cascaral 2008

K Ammari, E Crépau 2018

E Cerpa, E Crépau 2020

J Pava, M Cavalcante 2021

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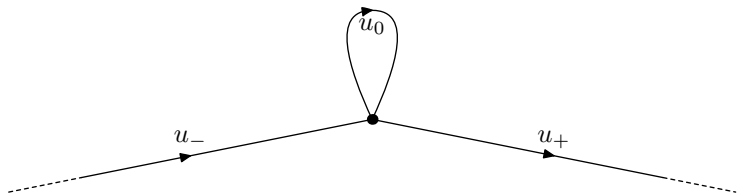
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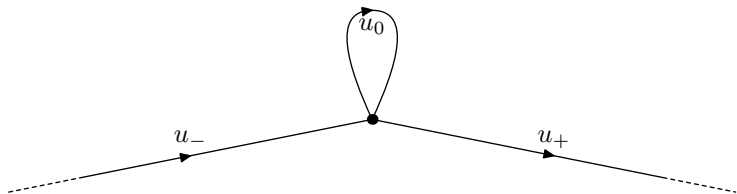
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So simplify by taking small amplitude linearization.

IKdV on the line with a defect

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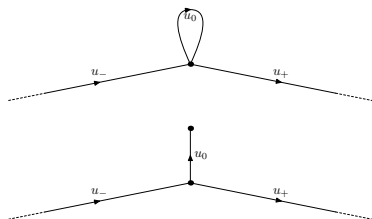
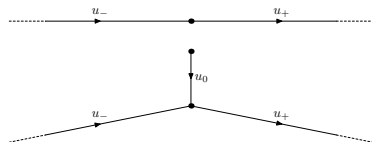
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Metric graph domains like



Wellposedness criteria on VC for these and closely related problems:

C Schubert, C Seifert, J Voigt, M Waurick 2015

B Deconinck, N Sheils, D Smith 2016,

D Mugnolo, D Noja, C Seifert 2018, 2024

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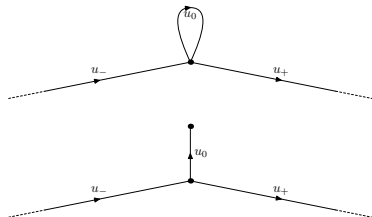
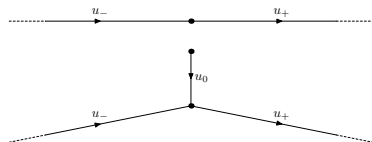
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Metric graph domains like



We aim to *solve* the problem, for select classes of VC.

Galilean transformation ($a = 0$) impossible on graph or domain with boundary.

Linear Schrödinger on a graph

The problem

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can be solved by following G Berkolaiko, P Kuchment *Introduction to quantum graphs* (2013) AMS:

- ① zoom in to each vertex so we get star graphs with semiinfinite edges
- ② transform to spectral variable in each star graph
- ③ solve scattering problem on each star graph
- ④ assemble scattering data for full problem from the star graphs' scattering data
- ⑤ transform back to coordinate space

Much more is known:

- bounds on eigenvalues
- effect of collapsing parallel edges into one
- trapped modes
- ...

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All this is formulated purely in spectral space. Requires transform pair.

IKdV on an interval

For interval $\Omega = (0, 1)$ or $\Omega = (0, \infty)$,

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$$\text{appropriate boundary conditions,} \quad t \geq 0, \quad (4.BC)$$

for wellposedness, we need 1 BC at $x = 0$ and decay as $x \rightarrow \infty$ or 2 BC at $x = 1$.

If we allow coupling (ie solve on a loop domain), then full wellposedness criteria are open.

Problem can be solved via unified transform method:

A Fokas *A unified approach to boundary value problems* (2008) CBMS-SIAM

B Deconinck, T Trogdon, V Vasani 2014 SIAM Review

But, generically, problem cannot be solved by real spectral transform or generalized Fourier series, because:

- problem is nonselfadjoint
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Contrast with linear Schrödinger problems, where transform / Fourier series works. So we abandon the “assemble scattering data” approach.

UTM on metric graph domains

We will adapt the unified transform method to problems on metric graphs.

Earlier work:

(line graph, heat) M Asvestas, A Sifalakis, E Papadopoulou, Y Saridakis, 2015

(line graph, heat) D Mantzavinos, G Papadomanolaki, A Sifalakis, Y Saridakis, 2016

(line graph, heat & linear Schrödinger) N Sheils, B Deconinck, ++ 2014–2020

(particular metric graphs, heat) N Sheils, D Smith, 2015

(line graph, IKdV) B Deconinck, N Sheils, D Smith, 2016,

(**star graph with 3 leads, IKdV**) Z Sobirov, M Eshimbetov, 2020,

(**metric graphs composed of finite intervals, general PDE**) S Aitzhan, S Bhandari, D Smith 2022,

(line graph, heat & IS, continuous limit) B Deconinck & M Farkas, 2023–2025

UTM overview

Based on the Fourier transform because that is the right transform to solve $u_t = u_{xxx}$ on $\mathbb{R}$.

Do UTM, which includes solving a linear system.

Obtain solution formula

$$u(x, t) = \mathcal{F}^{-1}[e^{-i\lambda^3 t} \mathcal{F}[U]](x)$$

a forward transform $\mathcal{F}$ applied to initial datum U and inverse transform $\mathcal{F}^{-1}$ which is a complex contour integral.

“Fokas transform” pair, which (weakly) diagonalises spatial differential operator.

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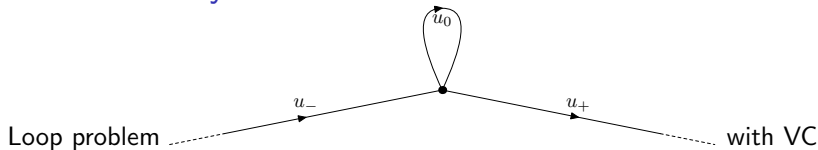
UTM linear systems

Finite interval problem linear system has matrix like $\begin{pmatrix} 1 & e^{-i\lambda} & e^{-i\lambda} \\ 1 & e^{-i\alpha\lambda} & \alpha e^{-i\alpha\lambda} \\ 1 & e^{-i\alpha^2\lambda} & \alpha^2 e^{-i\alpha^2\lambda} \end{pmatrix}$

Mismatch problem $\cdots \cdots \cdots \xrightarrow{u_-} \bullet \xrightarrow{u_+} \cdots \cdots \cdots$ with VC

$B_k \partial_x^k u_-(0^-, t) = \partial_x^k u_+(0^+, t)$ linear system has matrix $\begin{pmatrix} 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \\ B_2 & B_1 & B_0 \end{pmatrix}$

UTM linear systems

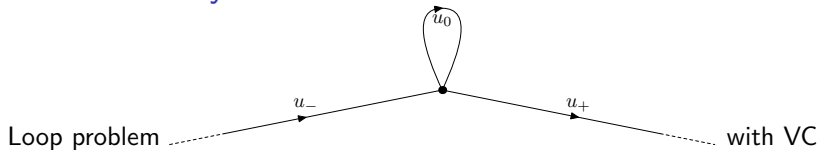


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linear system has matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \alpha^2 & \alpha \\ 1 & 0 & 0 & 0 & \alpha & \alpha^2 \\ 0 & 1 & 0 & 0 & -B_1 & B_0 \\ 0 & 0 & 1 & e^{-i\lambda} & (e^{-i\lambda}\beta_1 - b_1) & (e^{-i\lambda}\beta_0 - b_0) \\ 0 & 0 & 1 & e^{-i\alpha\lambda} & \alpha(e^{-i\alpha\lambda}\beta_1 - b_1) & \alpha^2(e^{-i\alpha\lambda}\beta_0 - b_0) \\ 0 & 0 & 1 & e^{-i\alpha^2\lambda} & \alpha^2(e^{-i\alpha^2\lambda}\beta_1 - b_1) & \alpha(e^{-i\alpha^2\lambda}\beta_0 - b_0) \end{pmatrix}.$$

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Source and sink problems:

have similarly complicated linear systems.

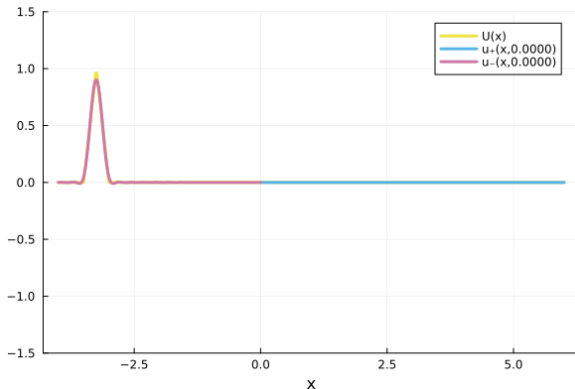


Solution of mismatch problem

Mismatch problem $\xrightarrow{u_-}$ $\bullet$ $\xrightarrow{u_+}$ with VC

$$B_k \partial_x^k u_-(0^-, t) = \partial_x^k u_+(0^+, t), \quad k \in \{0, 1, 2\}$$

for $(B_0, B_1, B_2) = (2.2, 2.0, 1.1)$ has solution

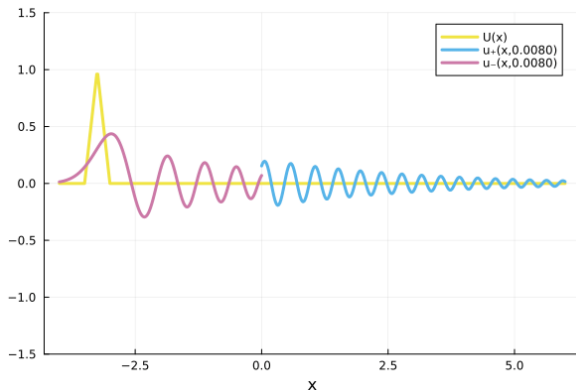


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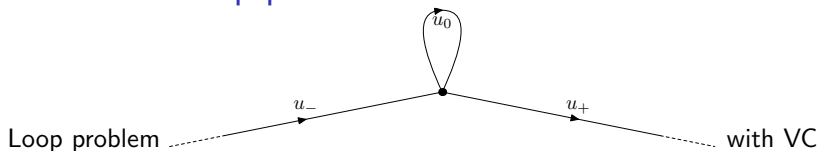
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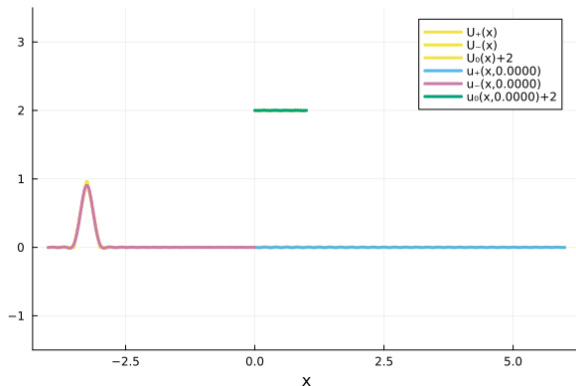
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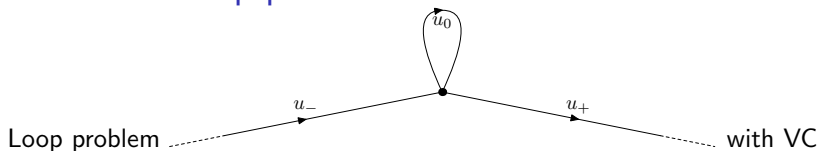
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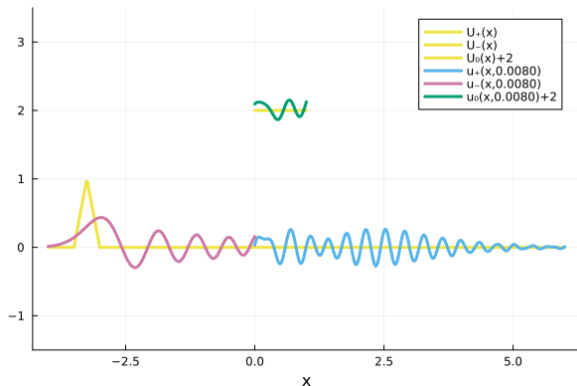
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Summary of progress

These solutions are presented in



D. A. Smith,

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arXiv preprint [arXiv:2503.12639](https://arxiv.org/abs/2503.12639), 1–53.

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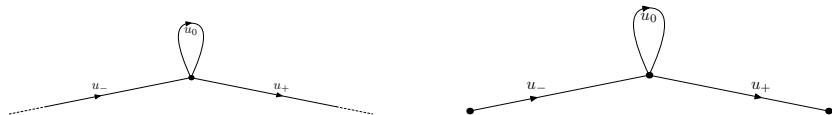


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And each new set of boundary conditions at $x = \pm 1$ require solving a new dimension 9 linear system.

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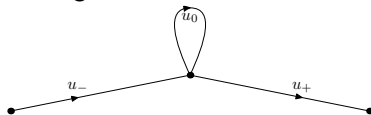
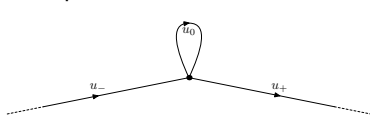
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So seek a new approach.

Inspiration from selfadjoint method

Aim: map solution on left domain to solution on right domain



In a selfadjoint problem, we could understand scattering by the defect vertex and scattering by each leaf vertex separately, then assemble into full scattering and reconstruct solution from there.

We can't use just the scattering data, as that doesn't contain enough information to reconstruct the solution.

UTM as a map of transform pairs: no lower order terms

Consider problems for (PDE) $[\partial_t - \partial_{xxx}]u(x, t) = 0$ and (IC) $u(x, t) = U(x)$.

Full line problem: (PDE), (IC), decay as $|x| \rightarrow \infty$.

Solved by Fourier transform: $u(x, t) = F^{-1}[e^{-i\lambda^3 t} F[U](\lambda)](x)$.

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Finite interval $(0, 1)$ problem: (PDE), (IC), (BC) at $x = 0, 1$.

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UTM is the process:

$$(F, F^{-1}) \mapsto (\mathcal{F}, \mathcal{F}^{-1})$$

where F, F^{-1} is the Fourier transform and its inverse, $\mathcal{F}, \mathcal{F}^{-1}$ are the “Fokas transform” pair that respect (BC).

Different (BC) yield different $\mathcal{F}, \mathcal{F}^{-1}$, but always constructed via UTM.

UTM as a map of transform pairs: with lower order terms

Let $\omega(\lambda) = \lambda^3 - a\lambda$ to allow for transport term.

Let ν be the biholomorphism with $\omega(\nu(\lambda)) = \lambda^3$, $\nu(\lambda) \sim \lambda$ as $\lambda \rightarrow \infty$.

Consider problems for (PDE) $[\partial_t + i\omega(-i\partial_x)]u(x, t) = 0$ and (IC) $u(x, t) = U(x)$.

Full line problem: (PDE), (IC), decay as $|x| \rightarrow \infty$.

Solved by Fourier-like transform: $u(x, t) = F^{-1}[e^{-i\lambda^3 t} F[U](\lambda)](x)$,

$$F[\phi](\lambda) = \int_{-\infty}^{\infty} e^{-i\nu(\lambda)x} \phi(x) dx, \quad F^{-1}[\hat{\phi}](x) = \frac{1}{2\pi} \int_{\Gamma} e^{i\nu(\lambda)x} \hat{\phi}(\lambda) \nu'(\lambda) d\lambda$$

for Γ the contour $(-\infty, \infty)$ but deformed away from 0 to keep ν defined.

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Finite interval $(0, 1)$ problem: (PDE), (IC), (BC) at $x = 0, 1$.

Solved by “Fokas transform”: $u(x, t) = \mathcal{F}^{-1}[e^{-i\lambda^3 t} \mathcal{F}[U](\lambda)](x)$.

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UTM is the process:

$$(F, F^{-1}) \mapsto (\mathcal{F}, \mathcal{F}^{-1})$$

where F, F^{-1} is the Fourier-like transform and its inverse, $\mathcal{F}, \mathcal{F}^{-1}$ are the “Fokas transform” pair that respect (BC).

Different (BC) yield different $\mathcal{F}, \mathcal{F}^{-1}$, but always constructed via UTM.

UTM as a map of transform pairs: way forward

So UTM does $(F, F^{-1}) \mapsto (\mathcal{F}, \mathcal{F}^{-1})$, where

(F, F^{-1}) is the transform pair for the PDE on the line,

$(\mathcal{F}, \mathcal{F}^{-1})$ is the transform pair for the same PDE but respecting BC.

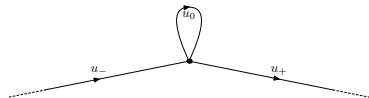
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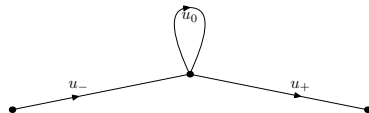
But we now have a transform pair for the “line” problem



Dimension 6 linear system that we already solved.

So use that as input (F, F^{-1}) for UTM.

Get output $(\mathcal{F}, \mathcal{F}^{-1})$ respecting BC for



Dimension 3 linear system.

but still respecting defect.

Serial UTM

- 1 Use the graph version of UTM to construct transform pair for “line with a defect” problem: $(F, F^{-1}) \mapsto (\mathcal{F}, \mathcal{F}^{-1})$
- 2 Use the finite interval version of UTM to construct transform pair for “finite interval with a defect” problem: $(\mathcal{F}, \mathcal{F}^{-1}) \mapsto (\mathcal{F}, \mathcal{F}^{-1})$.

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But 2 could be the graph version of UTM too.

Improved 2: From transform pairs for each vertex

$$(\mathcal{F}_{v_1}, \mathcal{F}_{v_1}^{-1}), (\mathcal{F}_{v_2}, \mathcal{F}_{v_2}^{-1}), \dots, (\mathcal{F}_{v_n}, \mathcal{F}_{v_n}^{-1})$$

construct new transform pair $(\mathcal{F}, \mathcal{F}^{-1})$ for whole graph, where 1 is done for each vertex.

At each step, the linear system's dimension is minimized.

Thanks

These slides

