

Linearized KdV on the line with a metric graph defect

Dave Smith

University of Newcastle, School of Information and Physical Sciences

FernUniversität
Hagen, Germany



KdV on the line with a defect

Rightward propagating waves in shallow water

$$u_t(x, t) - u_{xxx} + u_x + 6uu_x = 0 \quad x \in \mathbb{R} \setminus \{0\}, \quad t > 0, \quad (1.\text{PDE})$$

$$u(x, 0) = U(x) \quad x \in \mathbb{R} \setminus \{0\}, \quad (1.\text{IC})$$

$$3 \text{ conditions on } \partial_x^j u_-(0^-, t), \partial_x^j u_+(0^+, t), \quad t \geq 0. \quad (1.\text{VC})$$

Here interface (or vertex) condition (1.VC) represents a defect in the channel.



For “linearizable” VC, solution method could be adapted from V Caudrelier 2015.

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Closely related to the separated half line problems $x > 0$, $x < 0$ with each of (1.VC) specifying only $\partial_x^j u_-(0^-, t)$ or $\partial_x^j u_+(0^+, t)$.

Wellposedness requires:

- one $\partial_x^j u_-(0^-, t)$ condition for the $x < 0$ problem,
- two $\partial_x^j u_+(0^+, t)$ conditions for the $x > 0$ problem.

See many works of J Bona, B Y Zhang, ...

Half line solution for “linearizable” VC: A S Fokas 2002

KdV on the line with a more complicated defect

Rightward propagating waves in shallow water

$$u_t(x, t) - u_{xxx} + u_x + 6uu_x = 0 \quad x \in \text{edges}, \quad t > 0, \quad (2.\text{PDE})$$

$$u(x, 0) = U(x) \quad x \in \text{edges}, \quad (2.\text{IC})$$

$$\text{appropriate vertex conditions}, \quad t \geq 0. \quad (2.\text{VC})$$

Here (2.VC) represents a complex defect in the channel.



As before.

KdV on the line with a more complicated defect

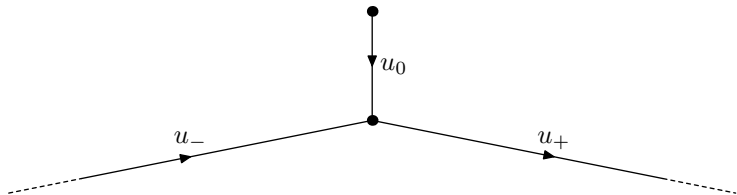
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Information is pushed into the line system at $x = 0$.

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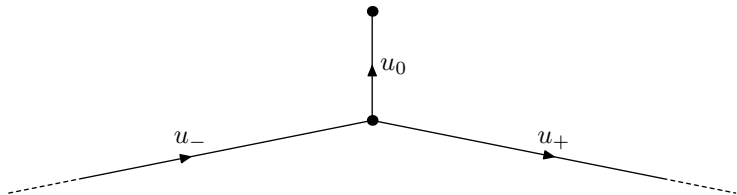
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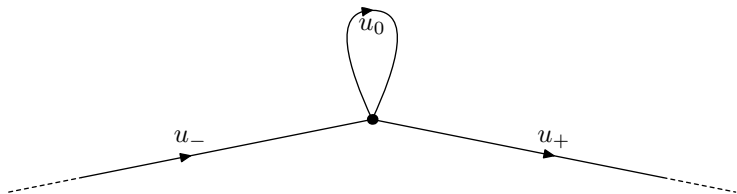
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A kind of delay in the line system at $x = 0$.

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Related wellposedness results:

Wellposedness for star graph problems with all edges semiinfinite:

M Cavalcante 2018.

Wellposedness for problems with all edges finite:

(for BBM) J Bona, R Cascaral 2008

K Ammari, E Crépau 2018

E Cerpa, E Crépau 2020

J Pava, M Cavalcante 2021

KdV on the line with a more complicated defect

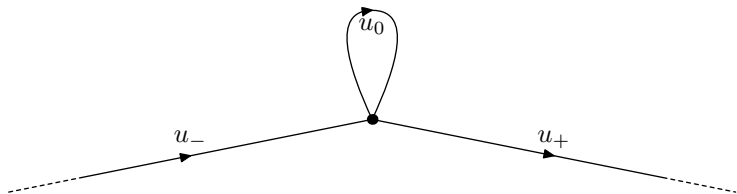
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May be solvable via nonlinear unified transform method for “linearizable” VC, but the finite intervals make it difficult.

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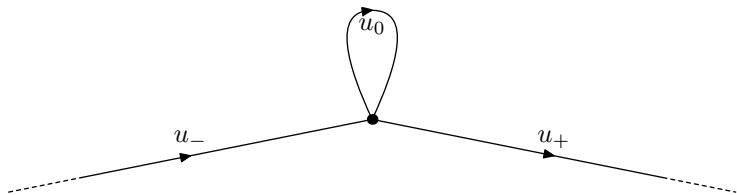
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May be solvable via nonlinear unified transform method for “linearizable” VC, but the finite intervals make it difficult.

So simplify by taking small amplitude linearization.

IKdV on the line with a defect

$$u_t(x, t) = u_{xxx} + au_x$$

$$u(x, 0) = U(x)$$

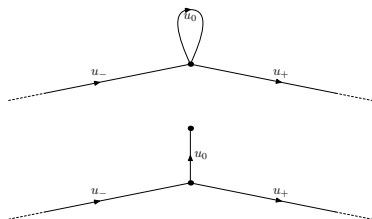
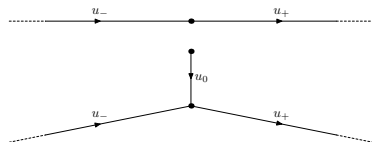
appropriate vertex conditions,

$$x \in \text{edges}, \quad t > 0, \quad (3.\text{PDE})$$

$$x \in \text{edges}, \quad (3.\text{IC})$$

$$t \geq 0. \quad (3.\text{VC})$$

Metric graph domains like



Wellposedness criteria on VC for these and closely related problems:

C Schubert, C Seifert, J Voigt, M Waurick 2015

B Deconinck, N Sheils, D Smith 2016,

D Mugnolo, D Noja, C Seifert 2018, 2024

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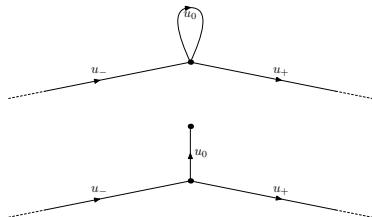
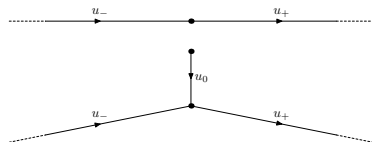
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Metric graph domains like



We aim to *solve* the problem, for select classes of VC.

Galilean transformation ($a = 0$) impossible on graph or domain with boundary.

Linear Schrödinger on a graph

The problem

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can be solved by following G Berkolaiko, P Kuchment *Introduction to quantum graphs* (2013) AMS:

- ① zoom in to each vertex so we get star graphs with semiinfinite edges
- ② transform to spectral variable in each star graph
- ③ solve scattering problem on each star graph
- ④ assemble scattering data for full problem from the star graphs' scattering data
- ⑤ transform back to coordinate space

Much more is known:

- bounds on eigenvalues
- effect of collapsing parallel edges into one
- trapped modes
- ...

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Much more is known:

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All this is formulated purely in spectral space. Requires transform pair.

IKdV on an interval

For interval $\Omega = (0, 1)$ or $\Omega = (0, \infty)$,

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$$\text{appropriate boundary conditions,} \quad t \geq 0, \quad (5.BC)$$

for wellposedness, we need 1 BC at $x = 0$ and decay as $x \rightarrow \infty$ or 2 BC at $x = 1$.

If we allow coupling (ie solve on a loop domain), then full wellposedness criteria are open.

Problem can be solved via unified transform method:

A Fokas *A unified approach to boundary value problems* (2008) CBMS-SIAM

B Deconinck, T Trogdon, V Vasan 2014 SIAM Review

But, generically, problem cannot be solved by real spectral transform or generalized Fourier series, because:

- problem is nonselfadjoint
- eigenfunctions of finite interval spatial differential operator are not complete (D Jackson 1915)

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Contrast with linear Schrödinger problems, where transform / Fourier series works. So we abandon the “assemble scattering data” approach.

UTM on graph domains

We will adapt the unified transform method to problems on graphs.

Earlier work:

(line graph, heat) M Asvestas, A Sifalakis, E Papadopoulou, Y Saridakis, 2015

(line graph, heat) D Mantzavinos, G Papadomanolaki, A Sifalakis, Y Saridakis, 2016

(line graph, heat & linear Schrödinger) N Sheils, B Deconinck, ++ 2014–2020

(networks, heat) N Sheils, D Smith, 2015

(line graph, IKdV) B Deconinck, N Sheils, D Smith, 2016,

(star graph with 3 leads, IKdV) Z Sobirov, M Eshimbetov, 2020,

(networks of finite intervals, general PDE) S Aitzhan, S Bhandari, D Smith 2022,

(line graph, heat & IS, continuous limit) B Deconinck & M Farkas, 2023–2025

Note also works by A Chatziafratas on UTM approach to existence & wellposedness, but on intervals.

These results have not been adapted to problems on line graphs or networks.

UTM overview

Based on the Fourier transform because that is the right transform to solve $u_t = u_{xxx}$ on $\mathbb{R}$.

Do UTM, which includes solving a linear system.

Obtain solution formula

$$u(x, t) = \mathcal{F}^{-1}[e^{-i\lambda^3 t} \mathcal{F}[U]](x)$$

a forward transform $\mathcal{F}$ applied to initial datum U and inverse transform $\mathcal{F}^{-1}$ which is a complex contour integral.

“Fokas transform” pair, which (weakly) diagonalises spatial differential operator.

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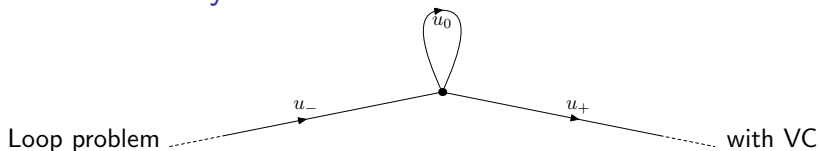
UTM linear systems

Finite interval problem linear system has matrix like $\begin{pmatrix} 1 & e^{-i\lambda} & e^{-i\lambda} \\ 1 & e^{-i\alpha\lambda} & \alpha e^{-i\alpha\lambda} \\ 1 & e^{-i\alpha^2\lambda} & \alpha^2 e^{-i\alpha^2\lambda} \end{pmatrix}$

Mismatch problem $\cdots \cdots \cdots \xrightarrow{u_-} \bullet \xrightarrow{u_+} \cdots \cdots \cdots$ with VC

$B_k \partial_x^k u_-(0^-, t) = \partial_x^k u_+(0^+, t)$ linear system has matrix $\begin{pmatrix} 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \\ B_2 & B_1 & B_0 \end{pmatrix}$

UTM linear systems

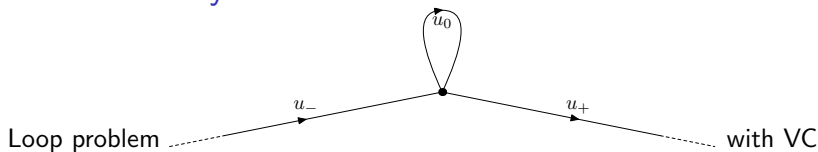


$$\partial_x^k u_-(0^-, t) = b_k^{-1} \partial_x^k u_0(0^+, t) = \beta_k^{-1} \partial_x^k u_0(1^-, t) = B_k^{-1} \partial_x^k u_+(0^+, t), \quad k \in \{0, 1\}$$

linear system has matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \alpha^2 & \alpha \\ 1 & 0 & 0 & 0 & \alpha & \alpha^2 \\ 0 & 1 & 0 & 0 & -B_1 & B_0 \\ 0 & 0 & 1 & e^{-i\lambda} & (e^{-i\lambda}\beta_1 - b_1) & (e^{-i\lambda}\beta_0 - b_0) \\ 0 & 0 & 1 & e^{-i\alpha\lambda} & \alpha(e^{-i\alpha\lambda}\beta_1 - b_1) & \alpha^2(e^{-i\alpha\lambda}\beta_0 - b_0) \\ 0 & 0 & 1 & e^{-i\alpha^2\lambda} & \alpha^2(e^{-i\alpha^2\lambda}\beta_1 - b_1) & \alpha(e^{-i\alpha^2\lambda}\beta_0 - b_0) \end{pmatrix}.$$

UTM linear systems



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Source and sink problems:



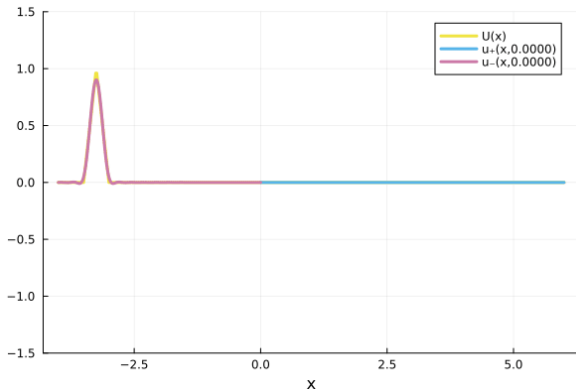
have similarly complicated linear systems.

Solution of mismatch problem

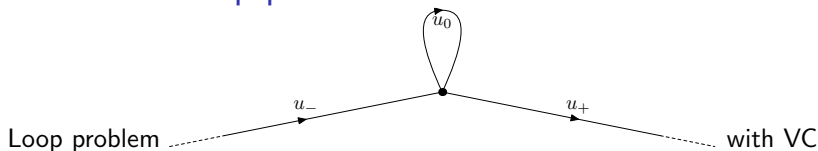
Mismatch problem $\xrightarrow{u_-}$ $\bullet$ $\xrightarrow{u_+}$ with VC

$$B_k \partial_x^k u_-(0^-, t) = \partial_x^k u_+(0^+, t), \quad k \in \{0, 1, 2\}$$

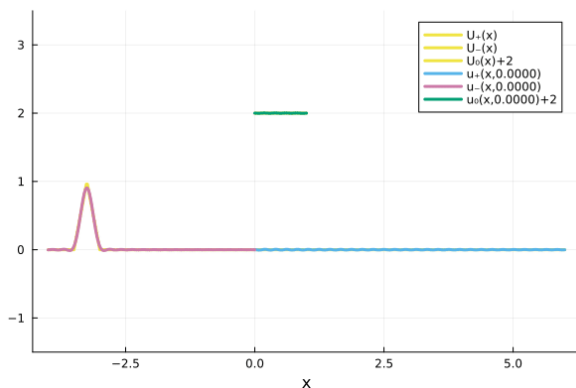
for $(B_0, B_1, B_2) = (2.2, 2.0, 1.1)$ has solution



Solution of loop problem



$\partial_x^k u_-(0^-, t) = b_k^{-1} \partial_x^k u_0(0^+, t) = \beta_k^{-1} \partial_x^k u_0(1^-, t) = B_k^{-1} \partial_x^k u_+(0^+, t), \quad k \in \{0, 1\},$
 for $(b_0, b_1, \beta_0, \beta_1, B_0, B_1) = (0.6, 0.7, 0.8, 2.1, 0.2, 3.0)$ has solution



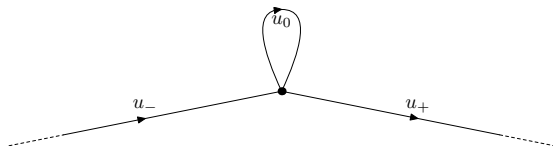
Mismatch wellposedness criteria

Mismatch problem $\cdots \xrightarrow{u_-} \bullet \xrightarrow{u_+} \cdots$ with VC

$$B_k \partial_x^k u_-(0^-, t) = \partial_x^k u_+(0^+, t), \quad k \in \{0, 1, 2\}$$

is wellposed provided $\sum_{k=0}^2 B_k \neq 0$.

Loop wellposedness criteria

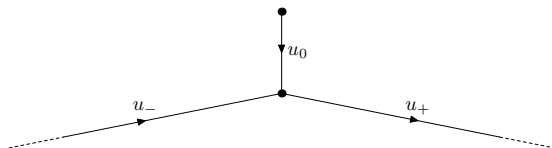


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is wellposed provided $|\beta_1 - \beta_0| > |\alpha b_1 - b_0|, |\alpha^2 b_1 - b_0|$.

Source wellposedness criteria

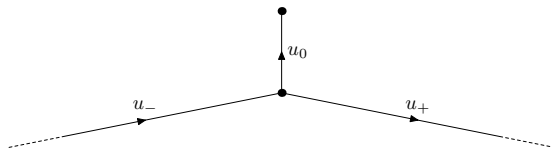


Source problem with VC

$$\begin{aligned}\partial_x^k u_-(0^-, t) &= \beta_k^{-1} \partial_x^k u_0(1^-, t) = B_k^{-1} \partial_x^k u_+(0^+, t), & k \in \{0, 1\}, \\ \partial_x^k u_0(0^+, t) &= H_k(t), & k \in \{0, 1\}\end{aligned}$$

is wellposed provided $\beta_0 \neq i\sqrt{3}\beta_1$.

Sink wellposedness criteria



Sink problem with VC

$$\begin{aligned}\partial_x^k u_-(0^-, t) &= b_k^{-1} \partial_x^k u_0(0^+, t) = B_k^{-1} \partial_x^k u_+(0^+, t), & k \in \{0, 1\}, \\ \partial_x^2 u_-(0^-, t) &= b_2 \partial_x^2 u_0(0^+, t) + B_2 \partial_x^2 u_+(0^+, t) \\ u_0(1^-, t) &= H(t)\end{aligned}$$

is wellposed provided $b_2(b_0 + b_1) - B_2(B_0 + B_1) \neq 1$.

Summary of progress

These solutions are presented in



D. A. Smith,

Linearized KdV on the line with a metric graph defect,
arXiv preprint [arXiv:2503.12639](https://arxiv.org/abs/2503.12639), 1–53.

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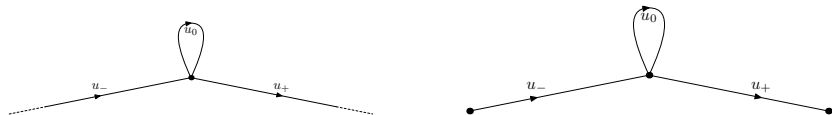


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Eg. loop problem linear system is dimension 6, but if the semiinfinite edges are replaced by finite edges, then it becomes dimension 9.

And each new set of boundary conditions at $x = \pm 1$ require solving a new dimension 9 linear system.

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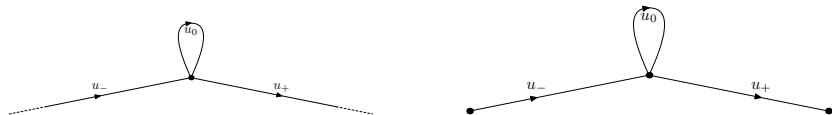


D. A. Smith,

Linearized KdV on the line with a metric graph defect,
[arXiv preprint arXiv:2503.12639](https://arxiv.org/abs/2503.12639), 1–53.

Strengths: Solves these problems, including transport term we omitted, gives first rigorous implementation of UTM existence result for network problems.

Weakness: Generalization to metric graphs with each extra (finite) edge adds 3 to dimension of linear system, which is undesirable.



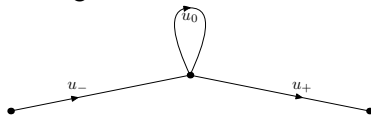
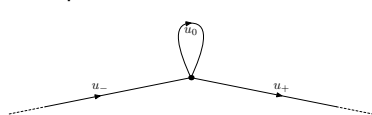
Eg. loop problem linear system is dimension 6, but if the semiinfinite edges are replaced by finite edges, then it becomes dimension 9.

And each new set of boundary conditions at $x = \pm 1$ require solving a new dimension 9 linear system.

So seek a new approach.

Inspiration from selfadjoint method

Aim: map solution on left domain to solution on right domain



In a selfadjoint problem, we could understand scattering by the defect vertex and scattering by each leaf vertex separately, then assemble into full scattering and reconstruct solution from there.

We can't use just the scattering data, as that doesn't contain enough information to reconstruct the solution.

UTM as a map of transform pairs: no lower order terms

Consider problems for (PDE) $[\partial_t - \partial_{xxx}]u(x, t) = 0$ and (IC) $u(x, t) = U(x)$.

Full line problem: (PDE), (IC), decay as $|x| \rightarrow \infty$.

Solved by Fourier transform: $u(x, t) = F^{-1}[e^{-i\lambda^3 t} F[U](\lambda)](x)$.

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UTM is the process:

input F, F^{-1} , output $\mathcal{F}, \mathcal{F}^{-1}$.

where F, F^{-1} is the Fourier transform and its inverse, $\mathcal{F}, \mathcal{F}^{-1}$ are the “Fokas transform” pair that respect (BC).

Different (BC) yield different $\mathcal{F}, \mathcal{F}^{-1}$, but always constructed via UTM.

UTM as a map of transform pairs: with lower order terms

Let $\omega(\lambda) = \lambda^3 - a\lambda$ to allow for transport term.

Let ν be the biholomorphism with $\omega(\nu(\lambda)) = \lambda^3$, $\nu(\lambda) \sim \lambda$ as $\lambda \rightarrow \infty$.

Consider problems for (PDE) $[\partial_t + i\omega(-i\partial_x)]u(x, t) = 0$ and (IC) $u(x, t) = U(x)$.

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for Γ the contour $(-\infty, \infty)$ but deformed away from 0 to keep ν defined.

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UTM as a map of transform pairs: way forward

So UTM does $(F, F^{-1}) \mapsto (\mathcal{F}, \mathcal{F}^{-1})$, where

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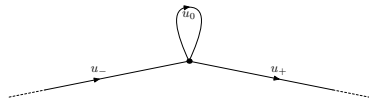
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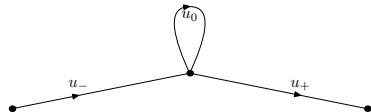
But we now have a transform pair for the “line” problem



Dimension 6 linear system that we already solved.

So use that as input (F, F^{-1}) for UTM.

Get output $(\mathcal{F}, \mathcal{F}^{-1})$ respecting BC for



Dimension 3 linear system.

but still respecting defect.

Serial UTM

- 1 Use the graph version of UTM to construct transform pair for “line with a defect” problem: $(F, F^{-1}) \mapsto (\mathcal{F}, \mathcal{F}^{-1})$
- 2 Use the finite interval version of UTM to construct transform pair for “finite interval with a defect” problem: $(\mathcal{F}, \mathcal{F}^{-1}) \mapsto (\mathcal{F}, \mathcal{F}^{-1})$.

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But 2 could be the graph version of UTM too.

Improved 2: From transform pairs for each vertex

$$(\mathcal{F}_{v_1}, \mathcal{F}_{v_1}^{-1}), (\mathcal{F}_{v_2}, \mathcal{F}_{v_2}^{-1}), \dots, (\mathcal{F}_{v_n}, \mathcal{F}_{v_n}^{-1})$$

construct new transform pair $(\mathcal{F}, \mathcal{F}^{-1})$ for whole graph, where 1 is done for each vertex.

At each step, the linear system's dimension is minimized.

Thanks

These slides

