

Fokas diagonalization

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Monash Analysis Seminar



Motivating problems: full line heat

$$\partial_t q(x, t) - \partial_{xx} q(x, t) = 0 \quad (x, t) \in \mathbb{R} \times (0, T), \quad (1.\text{PDE})$$

$$q(x, 0) = Q(x) \quad x \in \mathbb{R}, \quad (1.\text{IC})$$

$$q(\cdot, t) \in \mathcal{S}(\mathbb{R}) \quad t \in [0, T], \quad (1.\text{BC})$$

Apply exponential Fourier transform $F_{\mathbb{R}}[\phi](\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} \phi(x) dx$ to get

$$\begin{aligned} \partial_t F_{\mathbb{R}}[q](\lambda; t) &= F_{\mathbb{R}}[\partial_{xx} q](\lambda; t) \\ &= -\lambda^2 F_{\mathbb{R}}[q](\lambda; t), \end{aligned} \quad \text{by IBP.}$$

Solve temporal ODE for $F_{\mathbb{R}}[q](\lambda; t) = e^{-\lambda^2 t} F_{\mathbb{R}}[Q](\lambda)$. Apply inverse Fourier transform:

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} F_{\mathbb{R}}[Q](\lambda) d\lambda.$$

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Argument uses 2 properties of the exponential Fourier transform:

- The transform diagonalizes of the spatial differential operator.
- The transform is invertible.

Motivating problems: half line Dirichlet heat

$$\partial_t q(x, t) - \partial_{xx} q(x, t) = 0 \quad (x, t) \in (0, \infty) \times (0, T), \quad (2.\text{PDE})$$

$$q(x, 0) = Q(x) \quad x \in [0, \infty), \quad (2.\text{IC})$$

$$q(0, t) = 0, \quad q(\cdot, t) \in \mathcal{S}[0, \infty) \quad t \in [0, T], \quad (2.\text{BC})$$

Apply Fourier sine transform $F_s[\phi](\lambda) = \int_0^\infty \sin(\lambda x) \phi(x) dx$ to get

$$\begin{aligned} \partial_t F_s[q](\lambda; t) &= F_s[\partial_{xx} q](\lambda; t) \\ &= -\lambda^2 F_s[q](\lambda; t), \end{aligned} \quad \text{by IBP.}$$

Solve temporal ODE for $F_s[q](\lambda; t) = e^{-\lambda^2 t} F_s[Q](\lambda)$. Apply inverse Fourier sine transform:

$$q(x, t) = F_s^{-1} \left[e^{-\lambda^2 t} F_s[Q](\lambda) \right] (x).$$

Motivating problems: half line Dirichlet heat

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$$q(x, t) = F_s^{-1} \left[e^{-\lambda^2 t} F_s[Q](\lambda) \right](x).$$

Argument uses 2 properties of the Fourier sine transform:

- The transform diagonalizes of the spatial differential operator.
- The transform is invertible.

Motivating problems: finite interval Dirichlet heat

$$\partial_t q(x, t) - \partial_{xx} q(x, t) = 0 \quad (x, t) \in (0, 1) \times (0, T), \quad (3.\text{PDE})$$

$$q(x, 0) = Q(x) \quad x \in [0, 1], \quad (3.\text{IC})$$

$$q(0, t) = 0, \quad q(1, t) = 0 \quad t \in [0, T], \quad (3.\text{BC})$$

Apply finite interval Fourier sine transform:

$$F_{\text{s ser}}[\phi](j) = 2 \int_0^1 \phi(x) \sin(j\pi x) dx, \quad F_{\text{s ser}}^{-1}[(C_j)_{j \in \mathbb{N}}](x) = \sum_{j=1}^{\infty} C_j \sin(j\pi x).$$

to get,

$$\begin{aligned} \partial_t F_{\text{s ser}}[q](j; t) &= F_{\text{s ser}}[\partial_{xx} q](j; t), & j \in \mathbb{N} \\ &= -(j\pi)^2 F_{\text{s ser}}[q](j; t), & \text{by IBP.} \end{aligned}$$

Solve temporal ODE for $F_{\text{s ser}}[q](j; t) = e^{-(j\pi)^2 t} F_{\text{s ser}}[Q](j; t)$. Apply inverse finite interval Fourier sine transform:

$$q(x, t) = F_{\text{s ser}}^{-1} \left[e^{-(j\pi)^2 t} F_{\text{s}}[Q](j) \right] (x).$$

Aside: Kernel of the Fourier Sine transform

The finite interval Fourier sine transform can be viewed as an inner product

$$F_{\text{ser}}[\phi]\left(\frac{\lambda}{\pi}\right) = \langle \phi, 2 \sin(\bar{\lambda}x) \rangle$$

Observe that

$$2 \sin(\bar{\lambda}x) \in \text{span}\{e^{i\bar{\lambda}x}, e^{-i\bar{\lambda}x}\}$$

$$= \text{eigenspace for formal eigenvalue } \bar{\lambda}^2 \text{ of adjoint of } -\frac{d^2}{dx^2}.$$

The particular element of this eigenspace chosen (and the particular eigenvalue $\bar{\lambda}^2 = j^2\pi^2$) is fixed by the requirement that $2 \sin(\bar{\lambda}x)$ be a true eigenfunction of the adjoint of the finite interval Dirichlet heat operator.

But it will be helpful to think of it, less precisely, as just some formal eigenfunction from that formal eigenspace.

Motivating problems: observations

There is a spectral transform method broadly applicable to initial boundary value problems for constant coefficient linear evolution equations in $1 + 1$ d.

You need the right transform for the job in the sense that:

- The transform diagonalizes of the spatial differential operator.
- The transform is invertible.

The forward transform $\mathbf{F}[\phi]$ is composed (somehow) of inner products of ϕ with formal eigenfunctions of the adjoint spatial differential operator.

But how do you find the right transform?

Classical generalization

Use separation of variables (solve a Sturm-Liouville problem) to derive the right transform pair.

Difficulty: relies on eigenfunctions forming a basis of the space of initial data.

This matters because, eg., operator in

$$\begin{aligned} [\partial_t + \partial_x^3]q(x, t) &= 0 & (x, t) &\in (0, 1) \times (0, T), \\ q(x, 0) &= Q(x) & x &\in [0, 1], \\ q(0, t) = q(1, t) = q_x(1, t) &= 0 & t &\in [0, T]. \end{aligned}$$

does not have a complete system of eigenfunctions (D. Jackson 1915), but the problem is well posed.

Modern generalization

Use Fokas transform method. See works of Fokas, Pelloni, Mantzavinos, Deconinck, Trogdon, Vasan, Sheils. . .

Inverse transform is a contour integral. Forward transform accepts a continuous (complex contour) parameter.

Avoids difficulty of incompleteness of eigenfunctions.

Aim

Find a transform that will work for a given initial boundary value problem.

Aims

Find a transform that will work for a given initial boundary value problem.

Find a *very slightly* more general transform method that can be used with that transform to solve the initial boundary value problem.

Plan

A. Some illustrative examples.

- ① Half line Neumann problem for Airy equation.
- ② Finite interval problem for the heat equation with a nonlocal condition.
- ③ Half line Dirichlet problem for the linearized BBM equation.

B. Two point problems on the finite interval.

- ① Describe more efficient derivation of the Fokas transform pair.
- ② Describe (in what sense) the Fokas transform pair diagonalizes the spatial differential operator.
- ③ Solve IBVP with the Fokas transform pair.

C. Interface problems

- ① Finite networks & piecewise constant coefficients.
- ② Linearized KdV with a metric graph defect.

A1. Half line Neumann problem for Airy equation

Study problem

$$[\partial_t + \partial_x^3]q(x, t) = 0 \quad (x, t) \in (0, \infty) \times (0, T), \quad (4.PDE)$$

$$q(x, 0) = Q(x) \quad x \in [0, \infty), \quad (4.IC)$$

$$q_x(0, t) = 0, \quad q(\cdot, t) \in \mathcal{S}[0, \infty) \quad t \in [0, T]. \quad (4.BC)$$

Suppose we know a linear transform pair that obeys $\mathbf{F}^{-1}[\mathbf{F}[\phi]](x) = \phi(x)$ and ...

$$\mathbf{F}\left[\frac{d^3}{dx^3}\phi\right](\lambda) = -i\lambda^3\mathbf{F}[\phi](\lambda). \quad \mathbf{F}\left[\frac{d^3}{dx^3}\phi\right](\lambda) = -i\lambda^3\mathbf{F}[\phi](\lambda) + \mathbf{R}[\phi](\lambda).$$

Then (4.PDE) implies

$$\partial_t \mathbf{F}[q](\lambda; t) - i\lambda^3 \mathbf{F}[q](\lambda; t) = 0.$$

Solve temporal ODE for

$$\mathbf{F}[q](\lambda; t) = e^{i\lambda^3 t} \mathbf{F}[Q](\lambda)$$

and apply inverse transform:

$$q(x, t) = \mathbf{F}^{-1}\left[e^{i\lambda^3 t} \mathbf{F}[Q](\lambda)\right](x).$$

Then (4.PDE) implies

$$\partial_t \mathbf{F}[q](\lambda; t) - i\lambda^3 \mathbf{F}[q](\lambda; t) = -\mathbf{R}[q](\lambda; t).$$

Solve temporal ODE for

$$\mathbf{F}[q](\lambda; t) = e^{i\lambda^3 t} \mathbf{F}[Q](\lambda) - \int_0^t e^{i\lambda^3(t-s)} \mathbf{R}[q](\lambda; s) ds$$

and apply inverse transform:

$$q(x, t) = \mathbf{F}^{-1}\left[e^{i\lambda^3 t} \mathbf{F}[Q](\lambda)\right](x) - \mathbf{F}^{-1}\left[\int_0^t e^{i\lambda^3(t-s)} \mathbf{R}[q](\lambda; s) ds\right](x).$$

A1. Half line Neumann problem for Airy equation

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$$[\partial_t + \partial_x^3]q(x, t) = 0 \quad (x, t) \in (0, \infty) \times (0, T), \quad (4.\text{PDE})$$

$$q(x, 0) = Q(x) \quad x \in [0, \infty), \quad (4.\text{IC})$$

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$$\mathbf{F}\left[\frac{d^3}{dx^3}\phi\right](\lambda) = -i\lambda^3\mathbf{F}[\phi](\lambda).$$

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and apply inverse transform:

$$q(x, t) = \mathbf{F}^{-1}\left[e^{i\lambda^3 t} \mathbf{F}[Q](\lambda)\right](x).$$

and apply inverse transform:

$$\begin{aligned} q(x, t) &= \mathbf{F}^{-1}\left[e^{i\lambda^3 t} \mathbf{F}[Q](\lambda)\right](x) \\ &\quad - \mathbf{F}^{-1}\left[\int_0^t e^{i\lambda^3(t-s)} \mathbf{R}[q](\lambda; s) ds\right](x). \end{aligned}$$

A1. What is the transform pair?

We can solve the half line Neumann problem for Airy equation (4) using a linear transform pair that obeys $\mathbf{F}^{-1}[\mathbf{F}[\phi]](x) = \phi(x)$ and

$$\mathbf{F}\left[\frac{d^3}{dx^3}\phi\right](\lambda) = -i\lambda^3\mathbf{F}[\phi](\lambda) + \mathbf{R}[\phi](\lambda), \quad \mathbf{F}^{-1}\left[\int_0^t e^{i\lambda^3(t-s)}\mathbf{R}[q](\lambda;s)ds\right](x) = 0,$$

for q satisfying the boundary conditions.

The transform pair is, for $\alpha = e^{2\pi i/3}$ and $D^+ = \{\lambda \in \mathbb{C} : \frac{\pi}{3} < \arg(\lambda) < \frac{2\pi}{3}\}$,

$$\mathbf{F}[\phi](\lambda) = \begin{cases} \int_0^\infty \phi(y)e^{-i\lambda y} dy & \lambda \in \mathbb{R}, \\ \int_0^\infty \phi(y) \left[\alpha^2 e^{-i\alpha\lambda y} + \alpha e^{-i\alpha^2\lambda y} \right] dy & \lambda \in \partial D^+, \end{cases}$$

$$\mathbf{F}^{-1}[F](x) = \frac{1}{2\pi} \int_{\mathbb{R} \cup \partial D^+} e^{i\lambda x} F(\lambda) d\lambda \quad x \in [0, \infty).$$

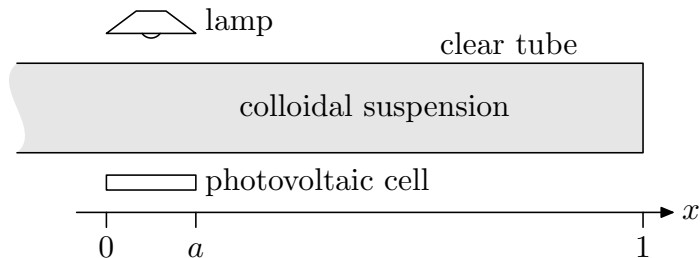
Proof sketch.

Invertability: Show $\int_{\partial D^+} e^{i\lambda x} F(\lambda) d\lambda = 0$, then Fourier inversion theorem.

Diagonalization: Show $\mathbf{R}[\phi](\lambda)$ is a polynomial of degree < 3 , with opposite sign on $\mathbb{R}$ and ∂D^+ .

Integrate by parts, use Jordan's lemma & Cauchy's theorem. □

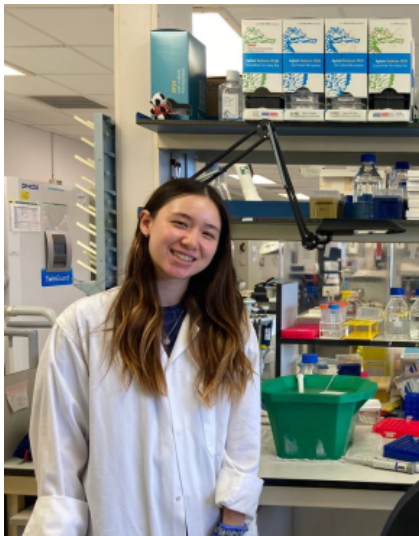
A2. Diffusion equation with a nonlocal condition



For q , concentration of dispersed substance, initial *boundary* value problem:

$$\begin{aligned} [\partial_t - \partial_x^2]q(x, t) &= 0 & (x, t) &\in (0, 1) \times (0, T), \\ q(x, 0) &= Q(x) & x &\in [0, 1], \\ q_x(1, t) &= 0 & t &\in [0, T], \\ q(0, t) &= \gamma(t) & t &\in [0, T], \end{aligned}$$

A2. Diffusion equation with a nonlocal condition



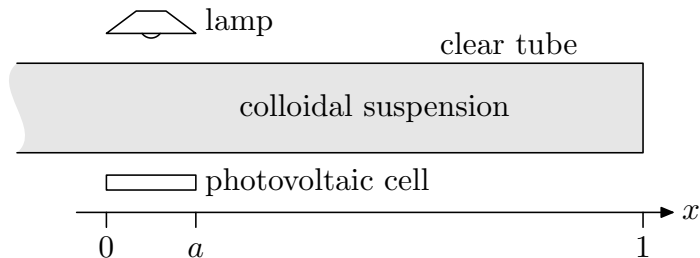
I can buy
expensive
instruments!

A2. Diffusion equation with a nonlocal condition



I can only afford
a finitely small
photovoltaic
cell!

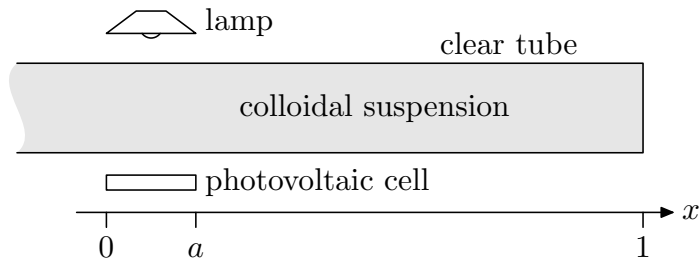
A2. Diffusion equation with a nonlocal condition



For q , concentration of dispersed substance, initial *nonlocal* value problem:

$$\begin{aligned} [\partial_t - \partial_x^2]q(x, t) &= 0 & (x, t) &\in (0, 1) \times (0, T), \\ q(x, 0) &= Q(x) & x &\in [0, 1], \\ q_x(1, t) &= 0 & t &\in [0, T], \\ \frac{1}{a} \int_0^a q(x, t) dx &= \hat{\gamma}(t) & t &\in [0, T], \end{aligned}$$

A2. Diffusion equation with a nonlocal condition



For q , concentration of dispersed substance, initial *nonlocal* value problem:

$$\begin{aligned} [\partial_t - \partial_x^2]q(x, t) &= 0 & (x, t) &\in (0, 1) \times (0, T), \\ q(x, 0) &= Q(x) & x &\in [0, 1], \\ q_x(1, t) &= 0 & t &\in [0, T], \\ \int_0^1 K(x)q(x, t)dx &= g(t) & t &\in [0, T], \end{aligned}$$

A2. Heat equation with a nonlocal condition

$$[\partial_t - \partial_x^2]q(x, t) = 0 \quad (x, t) \in (0, 1) \times (0, T), \quad (5.\text{PDE})$$

$$q(x, 0) = Q(x) \quad x \in [0, 1], \quad (5.\text{IC})$$

$$\int_0^1 K(y)q(y, t) \, dy = 0 = q_x(1, t) \quad t \in [0, T]. \quad (5.\text{BC})$$

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The diffusion equation with nonlocal data,
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 B. Normatov and D. A. Smith,
The Airy equation with nonlocal conditions,
Stud. Appl. Math. **152** 2 (2024), 543–567.

 D. A. Smith,
On some nonlocal diffusion problems, (in preparation).

A2. Heat equation with a nonlocal condition

$$[\partial_t - \partial_x^2]q(x, t) = 0 \quad (x, t) \in (0, 1) \times (0, T), \quad (5.\text{PDE})$$

$$q(x, 0) = Q(x) \quad x \in [0, 1], \quad (5.\text{IC})$$

$$\int_0^1 K(y)q(y, t) dy = 0 = q_x(1, t) \quad t \in [0, T]. \quad (5.\text{BC})$$

Suppose we know a linear transform pair that obeys $\mathbf{F}^{-1}[\mathbf{F}[\phi]](x) = \phi(x)$ and

$$\mathbf{F}\left[-\frac{d^2}{dx^2}\phi\right](\lambda) = \lambda^2 \mathbf{F}[\phi](\lambda) + \mathbf{R}[\phi](\lambda).$$

Then (5.PDE) implies

$$\partial_t \mathbf{F}[q](\lambda; t) + \lambda^2 \mathbf{F}[q](\lambda; t) = -\mathbf{R}[q](\lambda; t).$$

Solve temporal ODE for

$$\mathbf{F}[q](\lambda; t) = e^{-\lambda^2 t} \mathbf{F}[Q](\lambda) - \int_0^t e^{-\lambda^2(t-s)} \mathbf{R}[q](\lambda; s) ds$$

and apply inverse transform:

$$q(x, t) = \mathbf{F}^{-1}\left[e^{-\lambda^2 t} \mathbf{F}[Q](\lambda)\right](x) - \mathbf{F}^{-1}\left[\int_0^t e^{-\lambda^2(t-s)} \mathbf{R}[q](\lambda; s) ds\right](x).$$

A2. What is the transform pair?

We can solve the finite interval problem for the heat equation with a nonlocal condition (5) using a linear transform pair that obeys $\mathbf{F}^{-1}[\mathbf{F}[\phi]](x) = \phi(x)$ and

$$\mathbf{F}\left[-\frac{d^2}{dx^2}\phi\right](\lambda) = \lambda^2 \mathbf{F}[\phi](\lambda) + \mathbf{R}[\phi](\lambda), \quad \mathbf{F}^{-1}\left[\int_0^t e^{-\lambda^2(t-s)} \mathbf{R}[q](\lambda; s) ds\right](x) = 0,$$

for q satisfying the boundary conditions.

The transform pair is, for $D_\rho^\pm = \{\lambda \in \mathbb{C} : \frac{\pi}{4} < \arg(\pm\lambda) < \frac{3\pi}{4}, |\lambda| > \rho\}$ and $\rho \gg 0$,

$$\mathbf{F}[\phi](\lambda) = \begin{cases} \int_0^1 e^{-i\lambda y} \phi(y) dy & \lambda \in \mathbb{R}, \\ -\zeta^+(\lambda; \phi)/\Delta(\lambda) & \lambda \in \partial D_\rho^+, \\ -e^{-i\lambda} \zeta^-(\lambda; \phi)/\Delta(\lambda) & \lambda \in \partial D_\rho^-, \end{cases}$$

$$\mathbf{F}^{-1}[F](x) = \frac{1}{2\pi} \int_{\mathbb{R} \cup D_\rho^+ \cup D_\rho^-} e^{i\lambda x} F(\lambda) d\lambda \quad x \in [0, 1],$$

where

$$\Delta(\lambda) = \int_0^1 K(y) \cos([1-y]\lambda) dy,$$

$$\zeta^+(\lambda; \phi) = \int_0^1 K(y) \cos([1-y]\lambda) \int_0^y e^{-i\lambda z} \phi(z) dz dy + \int_0^1 K(y) e^{-i\lambda y} \int_y^1 \cos([1-z]\lambda) \phi(z) dz dy,$$

$$\zeta^-(\lambda; \phi) = \int_0^1 K(y) \int_y^1 \sin([z-y]\lambda) \phi(z) dz dy.$$

A3. Half line Dirichlet problem for linearized BBM

Study problem

$$[\partial_t(1 - \partial_x^2) + \partial_x]q(x, t) = 0 \quad (x, t) \in (0, \infty) \times (0, T), \quad (6.PDE)$$

$$q(x, 0) = Q(x) \quad x \in [0, \infty), \quad (6.IC)$$

$$q(0, t) = 0 \quad t \in [0, T]. \quad (6.BC)$$

More generally,

$$[\partial_t M + L]q(\cdot, t) = 0,$$

for spatial differential operators L, M having the same domain: smooth functions obeying the boundary conditions.

We need a transform pair that obeys $\mathbf{F}^{-1}[\mathbf{F}[\phi]](x) = \phi(x)$ and

$$\mathbf{F}[L\phi](\lambda) = \omega_L(\lambda)\mathbf{F}[\phi](\lambda) + \mathbf{R}_L[\phi](\lambda),$$

$$\mathbf{F}[M\phi](\lambda) = \omega_M(\lambda)\mathbf{F}[\phi](\lambda) + \mathbf{R}_M[\phi](\lambda),$$

with

$$\mathbf{F}^{-1} \left[\int_0^t e^{-\frac{\omega_L(\lambda)}{\omega_M(\lambda)}(t-s)} \frac{\mathbf{R}_L[q](\lambda; s) + \mathbf{R}_M[q_t](\lambda; s)}{\omega_M(\lambda)} ds \right] (x) = 0.$$

A3. Transform method with mixed partials

Using our transform on (6.PDE), we get

$$0 = \partial_t \mathbf{F} [Mq] (\lambda; t) + \mathbf{F} [Lq] (\lambda; t).$$

The diagonalization with remainder implies

$$0 = \partial_t (\omega_M(\lambda) \mathbf{F} [q] (\lambda; t) + \mathbf{R}_M [q] (\lambda; t)) + \omega_L(\lambda) \mathbf{F} [q] (\lambda; t) + \mathbf{R}_L [q] (\lambda; t).$$

Multiply by $e^{t\omega_L(\lambda)/\omega_M(\lambda)}/\omega_M(\lambda)$ and rearrange for

$$0 = \partial_t \left(e^{t\omega_L(\lambda)/\omega_M(\lambda)} \mathbf{F} [q] (\lambda; t) \right) + e^{t\omega_L(\lambda)/\omega_M(\lambda)} \frac{\mathbf{R}_L [q] (\lambda; t) + \mathbf{R}_M [q_t] (\lambda; t)}{\omega_M(\lambda)}.$$

Solve temporal ODE for

$$\mathbf{F} [q] (\lambda; t) = e^{-\frac{\omega_L(\lambda)}{\omega_M(\lambda)} t} \mathbf{F} [Q] (\lambda) - \int_0^t e^{-\frac{\omega_L(\lambda)}{\omega_M(\lambda)} (t-s)} \frac{\mathbf{R}_L [q] (\lambda; s) + \mathbf{R}_M [q_t] (\lambda; s)}{\omega_M(\lambda)} ds$$

and apply inverse transform:

$$q(x, t) = \mathbf{F}^{-1} \left[e^{-\frac{\omega_L(\lambda)}{\omega_M(\lambda)} t} \mathbf{F} [Q] (\lambda) \right] (x) - \mathbf{F}^{-1} \left[\int_0^t e^{-\frac{\omega_L(\lambda)}{\omega_M(\lambda)} (t-s)} \frac{\mathbf{R}_L [q] (\lambda; s) + \mathbf{R}_M [q_t] (\lambda; s)}{\omega_M(\lambda)} ds \right] (x).$$

A. Conclusions

If you can find a transform pair which diagonalizes the spatial part of the differential operator in the right way, then the same transform method will always work.

This can be extended to problems with mixed partials, with the natural generalization of the transform method.

The criteria for success of the transform method show how the transform pair must diagonalize the differential operator.

B3. Solve IBVP with Fokas transform pair: dream scenario

Differential operator & boundary conditions:

$$L\phi = \omega(-i\partial_x)\phi, \quad 0 = B_k\phi := \sum_{j=1}^n b_{kj}\phi^{(j-1)}(0) + \beta_{kj}\phi^{(j-1)}(1)$$

where ω monic of degree n and $k = 1, \dots, n$. Study problem

$$\partial_t q(x, t) + aLq(\cdot, t) = 0 \quad (x, t) \in (0, 1) \times (0, T), \quad (7.\text{PDE})$$

$$q(x, 0) = Q(x) \quad x \in [0, 1], \quad (7.\text{IC})$$

$$B_k q(\cdot, t) = 0 \quad t \in [0, T], \quad k = 1, \dots, n. \quad (7.\text{BC})$$

Suppose we know a transform pair that obeys $\mathbf{F}^{-1}[\mathbf{F}[\phi]](x) = \phi(x)$ and $\mathbf{F}[L\phi](\lambda) = \lambda^n \mathbf{F}[\phi](\lambda)$.
Then (7.PDE):

$$\partial_t \mathbf{F}[q](\lambda; t) = -a\lambda^n \mathbf{F}[q](\lambda; t).$$

Solve temporal ODE for $\mathbf{F}[q](\lambda; t) = e^{-a\lambda^n t} \mathbf{F}[Q](\lambda)$ and apply inverse transform:

$$q(x, t) = \mathbf{F}^{-1} \left[e^{-a\lambda^n t} \mathbf{F}[Q](\lambda) \right] (x).$$

B3. Solve IBVP with Fokas transform pair: real scenario

Suppose $\mathbf{F}^{-1}[\mathbf{F}[\phi]](x) = \phi(x)$ holds, but $\mathbf{F}[L\phi](\lambda) = \lambda^n \mathbf{F}[\phi](\lambda)$ is replaced by

$$\mathbf{F}[L\phi](\lambda) = \lambda^n \mathbf{F}[\phi](\lambda) + \mathbf{R}[\phi](\lambda),$$

where, for appropriate functions $q(\cdot, t)$ obeying boundary conditions,

$$\mathbf{F}^{-1} \left[\int_0^t e^{a\lambda^n(s-t)} \mathbf{R}[q](\lambda; s) ds \right] (x) = 0.$$

Then (7.PDE) implies

$$\partial_t \mathbf{F}[q](\lambda; t) = -a\lambda^n \mathbf{F}[q](\lambda; t) - a\mathbf{R}[q](\lambda; t).$$

Solve temporal ODE for

$$\mathbf{F}[q](\lambda; t) = e^{-a\lambda^n t} \mathbf{F}[Q](\lambda) - a e^{-a\lambda^n t} \int_0^t e^{a\lambda^n s} \mathbf{R}[q](\lambda; s) ds$$

and apply inverse transform for

$$q(x, t) = \mathbf{F}^{-1} \left[e^{-a\lambda^n t} \mathbf{F}[Q](\lambda) \right] (x) - a \mathbf{F}^{-1} \left[\int_0^t e^{a\lambda^n(s-t)} \mathbf{R}[q](\lambda; s) ds \right] (x).$$

B3. Solve IBVP with Fokas transform pair: observation

We do need invertability: $\mathbf{F}^{-1}[\mathbf{F}[\phi]](x) = \phi(x)$.

But true diagonalization, $\mathbf{F}[L\phi](\lambda) = \lambda^n \mathbf{F}[\phi](\lambda)$, is not required.

It is enough to have

$$\mathbf{F}[L\phi](\lambda) = \lambda^n \mathbf{F}[\phi](\lambda) + \mathbf{R}[\phi](\lambda),$$

where, for appropriate functions $q(\cdot, t)$ obeying boundary conditions,

$$\mathbf{F}^{-1} \left[\int_0^t e^{a\lambda^n(s-t)} \mathbf{R}[q](\lambda; s) ds \right] (x) = 0,$$

for the transform method to work.

B2. Diagonalization how?

Theorem

There exists a remainder transform $\mathbf{R}$ such that, for all ϕ satisfying the boundary conditions,

$$\mathbf{F}[L\phi](\lambda) = \lambda^n \mathbf{F}[\phi](\lambda) + \mathbf{R}[\phi](\lambda) \quad (8a)$$

and, if $q : [0, 1] \times [0, T] \rightarrow \mathbb{C}$ is such that, for all $t \in [0, T]$, $q(\cdot, t)$ satisfying the boundary conditions and, for all $j \in \{0, 1, \dots, n-1\}$, uniformly for all $x \in [0, 1]$, $\partial_x^j q(x, \cdot)$ is a function of bounded variation, then, for all $t \in [0, T]$ and all $x \in (0, 1)$,

$$\mathbf{F}^{-1} \left[\int_0^t e^{a\lambda^n(s-t)} \mathbf{R}[q](\lambda; s) ds \right] (x) = 0. \quad (8b)$$

Proof sketch.

Precisely define $\mathbf{R}[\phi](\lambda)$. Show $\mathbf{R}[\phi](\lambda)$ is analytic and $\mathcal{O}(|\lambda|^{n-1})$ as $\lambda \rightarrow \infty$, uniformly in $\arg(\lambda)$. To show (8b): integrate by parts to get $1/a\lambda^n$ factor so $\mathcal{O}(|\lambda|^{-1})$ overall. Use Jordan's lemma & Cauchy's theorem. $\square$

B1. Derivation of transform: biholomorphic map

Proposition

For sufficiently large $\mathcal{R} > 0$, there is a function ν which satisfies

$$\omega(\nu(\lambda)) = \lambda^n, \quad \lim_{\lambda \rightarrow \infty} \nu(\lambda)/\lambda = 1,$$

is biholomorphic outside the disc centered at zero with radius $\mathcal{R}$, and whose derivative approaches 1 as $\lambda \rightarrow \infty$, uniformly in the argument of λ .

Not a new theorem, but we give an elementary proof.

Allows us to think of the eigenvalues as λ^n instead of $\omega(\lambda)$, simplifying some arguments.

B1. Derivation of transform: formal transform pair

“Formal” because it works with the formal differential operator; it ignores the boundary conditions.

$$\mathcal{F}[\phi](\lambda) := \left\langle \phi, e^{i\nu(\lambda)\cdot} \right\rangle = \int_0^1 \phi(x) e^{-i\nu(\lambda)x} dx, \quad |\lambda| > \mathcal{R}, \quad (9a)$$

$$\mathcal{F}^{-1}[F](x) := \frac{1}{2\pi} \text{p.v.} \int_{\gamma} e^{i\nu(\lambda)x} \nu'(\lambda) F(\lambda) d\lambda, \quad (9b)$$

γ a perturbation of $\mathbb{R}$ to avoid the disc on which ν is undefined.

Theorem

Suppose that $\phi \in C^1[0, 1]$. Then, for all $x \in (0, 1)$, $\mathcal{F}^{-1}[\mathcal{F}[\phi]](x) = \phi(x)$.

Proof sketch.

Change of variables $k = \nu(\lambda)$ in usual exponential Fourier transform. □

B1. Derivation of transform: forward transform

Let $\alpha = \exp(2\pi i/n)$, so that $\omega(\nu(\alpha^j \lambda)) = (\alpha^j \lambda)^n = \lambda^n$, for $j = 1, 2, \dots, n$.

Build forward transform out of inner products with certain eigenfunctions of the adjoint formal differential operator. Recall that we expect

$$\mathbf{F}[\phi](\lambda) = \left\langle \phi, \text{certain linear combination of } e^{i\nu(\alpha^j \lambda) \cdot} \right\rangle.$$

Define the adjoint $L^* \psi = \bar{\omega}(-i\partial_x)\psi$ so that $\langle L\phi, \psi \rangle = \langle \phi, L^* \psi \rangle$ with adjoint boundary forms B_k^* . Let M be the characteristic determinant of L^* ,

$$M_{j\ k}(\lambda) = \overline{B_k^* e^{i\nu(\alpha^{j-1} \lambda) \cdot}}, \quad \Delta(\lambda) = \det M(\lambda).$$

Let $\mathbf{M}_{j\ k}(\lambda)$ be the critical minors of M and let $M^\pm$ be the part of M depending on just the left/right boundary. Then $\mathcal{F}[\phi](\lambda) = \mathcal{F}^\pm[\phi](\lambda)$ for $\lambda \in \mathbb{C}^\pm$,

$$\mathcal{F}^+[\phi](\lambda) = \frac{\nu'(\lambda)}{2\pi\Delta(\lambda)} \sum_{j=1}^n \sum_{k=1}^n (-1)^{(n-1)(j+k)} \det \mathbf{M}_{j\ k}(\lambda) M_{1\ k}^+(\lambda) \int_0^1 \phi(x) e^{-i\nu(\alpha^{j-1} \lambda)x} dx,$$

$$\mathcal{F}^-[\phi](\lambda) = \frac{-\nu'(\lambda)}{2\pi\Delta(\lambda)} \sum_{j=1}^n \sum_{k=1}^n (-1)^{(n-1)(j+k)} \det \mathbf{M}_{j\ k}(\lambda) M_{1\ k}^-(\lambda) \int_0^1 \phi(x) e^{-i\nu(\alpha^{j-1} \lambda)x} dx.$$

B1. Derivation of transform: inverse transform

Inverse transform is defined by

$$\mathbf{F}^{-1}[F](x) := \text{p.v.} \int_{\Gamma} e^{i\nu(\lambda)x} F(\lambda) d\lambda,$$

where Γ is a perturbation of $\{\lambda \in \mathbb{C} : \text{Re}(\lambda^n) = 0\}$.

Theorem

Suppose $\phi \in C^1[0, 1]$ and **certain integrals evaluate to 0**. Then, for all $x \in (0, 1)$,
 $\mathbf{F}^{-1}[\mathbf{F}[\phi]](x) = \phi(x)$.

Proof sketch.

Contours chosen so that, under the assumption, much of the contour integrals cancels. Forward transform chosen so that remaining parts simplify to $\mathcal{F}^{-1}[\mathcal{F}[\phi]](x)$. Apply validity theorem of formal transform pair. $\square$

Assumption is not too restrictive. Requires eigenvalues aren't in the wrong places and certain ratios of analytic functions decay. Such assumptions *always* appear (or are proved) in the unified transform method.

B1. Derivation of transform: observations

The problem of deriving the Fokas transform pair is reduced to construction of the classical adjoint.

And that can be done algorithmically; see Coddington & Levinson book 1955.

A student, Linda Linfan Xiao, implemented it in julia as undergraduate thesis.

Another student, Sultan Aitzhan, reimplemented it in ApproxFun.

C1. Interface problems: statement of problem

Several copies of finite interval $[0, 1]$.

On each a different formal differential operator, all same degree.

Boundary conditions prescribe linear combinations of all boundary values from all intervals.

Defines L , a list differential operator. The initial interface value problem is

$$\partial_t q(x, t) + \mathbf{a} \circ Lq(x, t) = \mathbf{0} \quad (x, t) \in (0, 1) \times (0, T), \quad (10.\text{PDE})$$

$$q(x, 0) = Q(x) \quad x \in [0, 1], \quad (10.\text{IC})$$

$$q(\cdot, t) \text{ obeys the boundary conditions} \quad t \in [0, T], \quad (10.\text{BC})$$

where $q(\cdot, t)$, Q are lists of functions, one on each copy of spatial interval $[0, 1]$, and $\mathbf{a}$ a list of temporal coefficients, $\circ$ is entrywise multiplication.

Note: this also includes piecewise constant coefficient PDE.

See also Deconinck & Farkas (forthcoming).

C1. Interface problems: via usual Fokas method

See Sheils's work for how to do this, several in collaboration with Deconinck.

Arduous to derive D-to-N map, because you have to solve very large linear systems by hand.

C1. Interface problems: via new approach

Sultan Aitzhan & Sambhav Bhandari constructed the adjoint of the interface differential operator.

Sultan implemented in julia & ApproxFun.

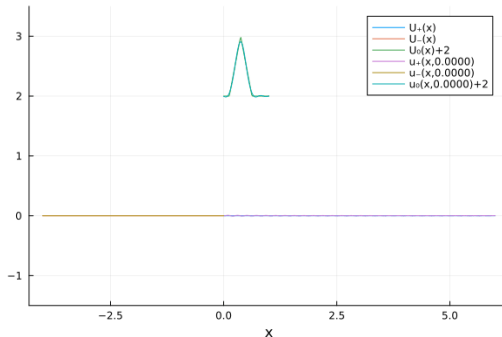
That gives you the definition of the transform pair.

Analogous diagonalization result holds.

So the same transform method can be used to solve the initial interface value problem.

C2. LKdV with a metric graph defect

Study linearized KdV on each edge of
with some vertex conditions. Solution:



D. A. Smith,

The linearized Korteweg-de Vries equation on the line with metric graph defects,

Stud. Appl. Math. **156** (2026), iss. 4, e70208, 1–51.

Conclusion

Fokas diagonalization is **not** like

$$\mathbf{F}[L\phi](\lambda) = \lambda^n \mathbf{F}[\phi](\lambda).$$

Instead, Fokas diagonalization is like

$$\mathbf{F}[L\phi](\lambda) = \lambda^n \mathbf{F}[\phi](\lambda) + \mathbf{R}[\phi](\lambda),$$

where, for appropriate functions $q(\cdot, t)$ obeying boundary conditions,

$$\mathbf{F}^{-1} \left[\int_0^t e^{a\lambda^n(s-t)} \mathbf{R}[q](\lambda; s) ds \right] (x) = 0.$$

But that is enough for the transform method to work, essentially unaltered.

We've got a quicker way to derive the transform, in terms of the characteristic determinant of the classical adjoint.

All this works for problems with interface or nonlocal conditions, or with mixed partials.

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AMSI Online course in Semester 1 2025–6, 2028–
<https://www.unifiedtransformlab.com/course/>

These slides



Research group <https://www.unifiedtransformlab.com>

